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JUNE, 1860.

EDITED BY

J. D. RUNKLE, A.M., A.A.S.

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THE

MATHEMATICAL MONTHLY.

Vol. II. . . . JUNE, 1860. . . . No. IX.

PRIZE PROBLEMS FOR STUDENTS.

I. The sum of the radii of the inscribed circle and of the circle touching the hypothenuse and extension of the two legs of a right-triangle is equal to the sum of the said legs.

II. The side of a regular octagon, inscribed in a circle whose radius is unity, is $\sqrt{(2 - \sqrt{2})}$, and its area is $2\sqrt{2}$. Required the proof.

III. The perpendicular to the normal of a parabola at its intersection with the principal axis, meets the curve in two points through which tangents are drawn; find the locus of their common point. — Communicated by Arthur W. Wright, New Haven, Ct.

IV. Required the radii of three equal circles tangent to each other, and cutting off equal areas from a given circle, so that the sum of these areas may be a maximum. — Communicated by G. B. Vose.

V. Two great circles are drawn at random on a sphere. What is the probability that their mutual inclination, taken less than 90°, will be contained between any given limits, as $n°$ and $m°$.

Solutions of these problems must be received by August 1, 1860.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VI., Vol. II.

The first Prize is awarded to George A. Osborne, Jr., Lawrence Scientific School, Cambridge, Mass.

The second Prize is awarded to David Trowbridge, Perry City, Schuyler Co., N. Y.

The third Prize is awarded to Miss Amanda M. Bennett, Saline, Washtenaw Co., Mich.

Prize Solution of Problem I.

By John J. Carter, Nunda Lit. Institute, N. Y.

Find x from the equation, $\sqrt[5]{(a+x)} + \sqrt[5]{(a-x)} = b$, by quadratics.

Raising the given equation to the fifth power and reducing, it becomes $2a + 5b^3(a^2 - x^2)^{\frac{1}{5}} - 5b(a^2 - x^2)^{\frac{2}{5}} = b^5$;

or $$(a^2 - x^2)^{\frac{2}{5}} - b^2(a^2 - x^2)^{\frac{1}{5}} = \frac{2a - b^5}{5b}.$$

This quadratic gives

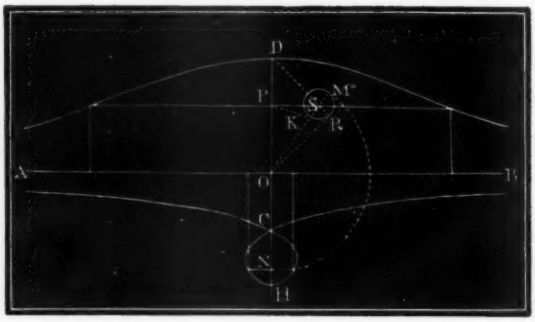
$$x = \pm \left\{ a^2 - \left(\frac{b^2}{2} \pm \sqrt{\frac{8a + b^5}{20b}} \right)^5 \right\}^{\frac{1}{2}}$$

Prize Solution of Problem III.

By George A. Osborne, Jr, Lawrence Scientific School, Cambridge, Mass.

Inscribe the maximum rectangle between the conchoid and its directrix.

Let AB be the directrix, and C the pole of the conchoid. Denote by a and b, respectively, the distances OD and OC. Then the equation of the curve referred to AB and CD as axis of X and Y, respectively, will be



(1) $$x^2 y^2 = (a^2 - y^2)(b + y)^2.$$

Since the curve is symmetrical with respect to the axis of Y, the inscribed rectangle will evidently be a maximum when xy or $x^2 y^2$ is a maximum.

Differentiating (1) gives

(2) $$D_y(x^2 y^2) = 2(b + y)(a^2 - by - 2y^2)$$

(3) $$D_y^2(x^2 y^2) = 2(a^2 - b^2 - 6by - 6y^2).$$

Putting $D_y(x^2 y^2) = 0$ gives

(4) $$y + b = 0, \quad \text{or } a^2 - by - 2y^2 = 0.$$

It is evident that the value $y = -b$ cannot correspond to a maximum, since the rectangle then vanishes.

The roots of (4) are found to be

$$y_1 = -\frac{b}{4} + \sqrt{\left(\frac{b}{4}\right)^2 + \frac{a^2}{2}}, \quad \text{and } y_2 = -\frac{b}{4} - \sqrt{\left(\frac{b}{4}\right)^2 + \frac{a^2}{2}}.$$

Substituting in (3) the value of y^2 from (4) gives

$$D_y^2(x^2 y^2) = -2(2a^2 + b^2 + 3by).$$

Whatever the relative values of a and b, y_1 is essentially positive, and renders $D_y^2(x^2 y^2) < 0$.

Hence there will always be a maximum rectangle between the directrix and the *superior* branch of the conchoid. The negative root y_2 will render $D_y^2(x^2 y^2) < 0$, when $\frac{b}{4} + \sqrt{\left(\frac{b}{4}\right)^2 + \frac{a^2}{2}} < \frac{2a^2 + b^2}{3b}$, which condition is found by reduction to be identical with $b < a$.

Moreover, when $b < a$ the numerical value of y_2 is less than a and greater than b. It therefore lies between CO and HO.

Hence we conclude that when $b < a$, that is to say, when the inferior branch of the conchoid forms a loop CH around the axis of Y, a maximum rectangle may be inscribed between the directrix and the curve, whose lower base is inscribed in this loop; but when

$b > a$, or when this loop does not exist, there is no maximum rectangle for the inferior branch.

To construct these rectangles, draw OR and DR making with OD angles of 45°. Take $RS = \frac{b}{4}$ and describe the circle MRK. Making OP and ON equal respectively to OK and OM determines the position of the required rectangles. It will be readily seen that this is merely a geometrical construction of y_1 and y_2.

Prize Solution of Problem IV.

By George A. Osborne, Jr., Lawrence Scientific School, Cambridge, Mass.

Given a cask containing a gallons of wine. Through a cock at the bottom of the cask wine flows out at the rate of b gallons per minute, and through a hole at the top water flows in at the same rate. Supposing the water, as fast as it flows in, to mingle perfectly with the wine, how long before the quantities of wine and water in the cask will be equal? and how much wine will be left in the cask at the end of t minutes?

Let x denote the number of gallons of wine in the cask at the end of t minutes. Then a being the number of gallons of the mixture, which escapes at the rate of b gallons per minute, the wine escapes at the rate of $\frac{x}{a} b$ gallons per minute.

The quantity of wine which escapes during the time dt is $\frac{x}{a} b\, dt = -dx$ since x varies inversely as t.

From this equation we have by integration $\frac{b}{a} t = -\log x +$ constant. Since $x = a$ when $t = 0$, $\frac{b}{a} t = \log \frac{a}{x}$, or $e^{\frac{bt}{a}} = \frac{a}{x}$, where e is the base of Napier's Logarithms. This gives the quantity of wine in the cask at any time t.

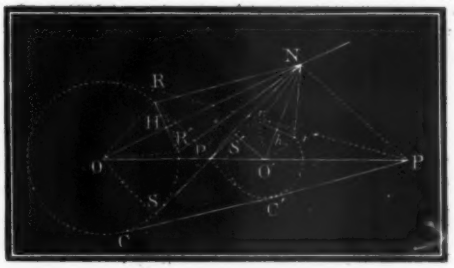
If the quantities of wine and water are equal $x = \frac{a}{2}$ and $t = \frac{a}{b} \log 2$.

Prize Solution of Problem V.

By Isaac H. Turrell, Mount Carmel, Ind.

Two circles being given in a plane, find geometrically the locus of the points from which chords of similar arcs in the two circles will be seen under the same angle, the chords being perpendicular to the lines of vision drawn through the centres of the given circles.

Let O and O' be the centres of the given circles. Draw the common tangents $C\,C'\,P$, $S\,P'\,S'$ cutting $O\,O'$ in P and P'. These points, which are called the external and internal centres of similitude of the two given circles, are evidently the only points on $O\,O'$, from which chords of similar arcs in the two circles will appear under the same angle. Let N be any other point from which the similar chords $R\,R'$, $r\,r'$, which are perpendicular to $N\,H\,O$, $N\,h\,O'$, respectively, will be seen under the equal angles $R\,N\,R'$ $r\,N\,r'$.



Since the triangles $O\,R\,N$, and $O'\,r\,N$ are similar

$$O\,N : O'\,N :: O\,R : O'\,r :: O\,P' : O'\,P' :: O\,P : O'\,P.$$

Therefore the line $N\,P'$ bisects the angle $O\,N\,O'$, and $N\,P$ bisects the supplement of $O'\,N\,O$. Therefore $P'\,N\,P$ is a right angle.

Consequently the required locus is a circle described on $P\,P'$ as a diameter.

Cor. — If the two given circles O and O' be considered spheres, the locus of the points will be a sphere described on $P\,P'$ as a diameter.

ON SPHERICAL ANALYSIS.

By George Eastwood, Saxonville, Mass.

[Continued from Page 272.]

Proposition V.

To find the equation of a great circle of the sphere which passes through one or more given points.

1. In Prop. III. we have found the equation of a great circle to be

$$y = \tau x + \beta.$$

That this circle may pass through a given point $x'y'$, we signify that it does so by saying that when the variable arc x becomes x', then y will become y'. Consequently the equation of the great circle becomes

$$y' = \tau x' + \beta.$$

Between these two equations, if we eliminate β, we have

$$(23) \quad y = \tau x + y' - \tau x', \qquad \text{or } y - y' = \tau(x - x'),$$

for the equation of a great circle which passes through a given point whose co-ordinates are $x'y'$.

When the given point $x'y'$ is situated at 90° from the origin, then $\tan x' = \tan y' = \infty$; and equation (23) is not applicable in this case. We can, however, overcome this apparent difficulty by introducing into our investigation the latitude of the point. For this purpose, let us consider the triangle YOX, which is formed by the two centres of projection and the origin. By supposition, the great circle AB cuts the side YX in the point P, the latitude of which we will designate by λ. Hence, by the theorem of spherical transversals, we have the ratios

$$\sin(90° - \beta) : \sin(\omega - \lambda) = \sin P : \sin B,$$
$$\sin\alpha : \sin\beta = \sin B : \sin A,$$
$$\sin\lambda : \sin(\alpha - 90°) = \sin A : \sin P,$$

which, by composition and expansion, become

$$\cos\beta \sin\alpha \sin\lambda = -\cos\alpha \sin\beta \sin(\omega - \lambda).$$

This equation may be expressed in the form

$$\frac{\sin\lambda}{\sin(\omega - \lambda)} \cdot \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\cos\beta}{\sin\beta} = -1;$$

or in this,

$$\frac{\beta}{\alpha} = -\frac{\sin\lambda}{\sin(\omega - \lambda)}.$$

Hence the equation of a great circle which passes through a given point, situated at 90° from the origin, will be truly defined by the equation

(24) $$y = \tau x + \beta,$$

in which τ represents $\frac{\sin \lambda}{\sin (\omega - \lambda)}$, or simply $\tan \lambda$, if the axes be taken rectangular.

Cor. — If any number of great circles have a common point of intersection, situated at 90° from the origin, the coefficient of x will be the same in all their equations.

2. If the proposed circle passes through the given points $x' y'$ and $x'' y''$, we must have by virtue of 1,

(*a*) $y = \tau x + \beta$, (*b*) $y' = \tau x' + \beta$, (*c*) $y'' = \tau x'' + \beta$.

Now (*a*) — (*b*) gives

(*d*) $$y - y' = \tau (x - x'),$$

and (*b*) — (*c*) gives

$$y' - y'' = \tau (x' - x''), \qquad \therefore \tau = \frac{y' - y''}{x' - x''}.$$

Substituting this value of τ in (*d*), we have finally

(25) $$y - y' = \frac{y' - y''}{x' - x''} (x - x').$$

This equation will assume different forms according to the particular position of the given points. For instance, if the point $x'' y''$ be on the axis of x, we have $y'' = 0$;

$$\therefore y - y' = \frac{y'}{x' - x''} (x - x').$$

If it be on the axis of y, then $x'' = 0$;

$$\therefore y - y' = \frac{y' - y''}{x'} (x - x').$$

If it be at the origin, then both x'' and $y'' = 0$;

$$\therefore y = \frac{y}{x'} x.$$

And if it be at 90° from the origin, then by virtue of (24)

$$y - y' = \tau (x - x').$$

If a great circle pass through three given points, the following relation must exist between the co-ordinates of those points;

(26) $(y' x'' - x' y'') - (y' x''' - x' y''') + (y'' x''' - x'' y''') = 0.$

[To be continued.]

THE ECONOMY AND SYMMETRY OF THE HONEY-BEES' CELLS.

By Chauncey Wright, Nautical Almanac Office, Cambridge, Mass.

The economical characteristics of the honey-cell have claimed so great attention from mathematicians, that other and even more prominent properties of its structure, though not unnoticed, have received too little attention. Psychologists, accordingly, following the testimony of mathematicians, have treated of the instinct of bees as if economy were not simply the most important, but even the only useful and noticeable feature of the bees' architecture. In reconsidering therefore the geometrical properties of the hive-cell we have chiefly in view the inferences which may be drawn from them in regard to the nature and powers of the bee's instinct.

In this review we shall first recount the reasoning *a priori* by which the structure of the hive-cell is deduced from considerations of rational economy, and then, in the second place, deduce the same results from considerations of symmetry applied to those modifications of the simple nest which are required by simple economy.

It is necessary here to premise a division of economy into two species, which we may denominate rational and sensible, or the

economy of forethought and simple economy; that which forestalls waste, and that which remedies waste or simply saves. The importance of this distinction will be apparent when, in the third place, we consider the two previous lines of argument in their relation to the faculties of instinct.

1. It is a proposition of elementary geometry that of all polygons or rectilinear plane figures with a given number of sides and a given perimeter, that figure contains the greatest area which is regular, or both equilateral and equiangular; and hence, that regular prisms have the greatest solid contents for a given convex surface. Again it is proved that a given perimeter encloses the greatest area in that regular polygon which has the greatest number of sides, so that among plane figures the circle has the greatest area for a given perimeter; and hence, also, the cylinder has the greatest solid contents among prismatic bodies for a given convex surface.

If now we seek among all regular polygons, or prisms, for those which are capable of dividing space into equal and similar parts without interstices, we readily perceive that the angles of such a figure must be aliquot parts of the circle or of four right angles. All the angles of any such figure are equal to twice as many right angles as the figure has sides minus four right angles, or if n be the number of sides, the sum of all the angles is $(2n - 4)$ right angles; hence each angle of a regular figure must bear the ratio $\frac{2n-4}{n}$ to a right angle, and as 4 right angles must be divisible by each of these angles, we have $\frac{4n}{2n-4}$, or $\frac{2n}{n-2}$ = an integer. But this number is equal to $2 + \frac{4}{n-2}$; hence $(n - 2)$ must be a divisor of 4, that is, either 1, 2, or 4, and n must therefore be either 3, 4, or 6; so that triangular, quadrilateral, and hexagonal regular polygons and prisms are the only ones which can collectively fill space without interstices. Now of these the one which has the greatest num-

ber of sides is the most economical. Hence the partition into regular hexagonal prisms is the most economical division of space in respect to two dimensions; but the third dimension is left undetermined.

If we suppose a hollow space, bounded by the surface of a regular hexagonal prism, to be open at one end, but closed at the other, so as to form a cell, it is clear that the base of this cell is most economized, if, like the sides, it be made also a boundary to another cell, or partially to several other cells. Hence, if two series of hexagonal cells, opening in opposite directions, have common bases, they present another feature of economy which is also exhibited in the form of the honeycomb. To this extent the bees' economy was known to the ancients.

There are two forms in which the bases of the cells might be fashioned so as to fit them as bases also of the opposite cells in the comb. They might be either plane hexagons perpendicular to the sides of the cells, or be composed of three rhombs inclined to the sides of the cells, and forming a solid angle in the central line or axis of the cell. If in the first form all the bases be in the same plane the relative positions of cells on opposite sides of the comb are immaterial to economy. But as each angle of the regular hexagonal prism is $\frac{4}{3}$ of a right angle, or $\frac{1}{3}$ of the circle, three regular hexagonal prisms have one common corner or line of contact, and this common corner must, in the second form of the bases, be in the same line with the axis of an opposite cell, the base of which is therefore composed of three thirds of the three contiguous bases.

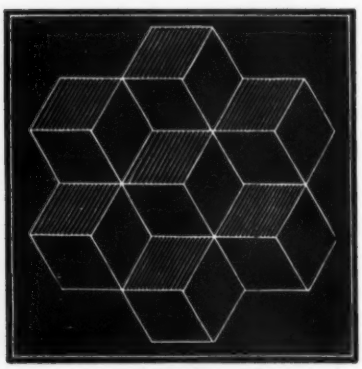


Fig. 1.

The dimensions of these rhombic segments of the bases vary with their inclinations to the sides and axes of the cells. One diagonal however is the same in position and length for all inclinations of the rhombs, while the other diagonal will vary with its inclination to the axis of the cell from the length of the shorter diagonal in the rhombs of a plane hexagonal base to any length whatever. This increase in the length of the variable diagonal increases the area of the rhombs, while the sides of the cells are more and more cut off and their area diminished as the rhombs become larger and more obliquely inclined.

In passing from the form of the plane hexagonal base, the sides at first diminish more rapidly than the rhombs increase, while the solid contents of the cell is the same for all the inclinations and corresponding forms of the rhombic segments of its base. It becomes a problem, therefore, of economy to determine that inclination and form of the rhombs which will give the least value to the sum of the areas in the sides and bases, or to the whole surface of the cell.

Instead of giving here the ordinary algebraic solution of this problem, which gives the same form to the rhombs as that determined by MARALDI'S measurements of the honey-cell, we shall proceed upon the second line of argument to deduce the same form from considerations of symmetry, and we will then prove by geometrical reasoning that this form is a minimum. We may remark here, however, that MARALDI determined by measurement, not merely the angles of the rhombs, but also a symmetry in the structure, which, as some geometricians have supposed, might have enabled him by calculation to give these angles, 109° 28′ and 70° 32′, with so great precision. The symmetry which MARALDI observed was the equality of the angles that the planes of the rhombs make with each other to the angle 120° which the sides of the cell make with each other, and also with the rhombs. This symmetry depends

directly on the regularity of all the solid angles in the cell, which MARALDI also pointed out. Indeed, if we regard only the angles which the *planes* make with each other, there is but one angle, 120°, in the whole structure. This one symmetry affords a perfect rule to the insect architect.

If we wish further to determine the angles which the *lines* make with each other, it is not necessary to make any direct measurements, but only to calculate the angles at the vertex of a triangular pyramid of which the sides make with each other the angle 120°. These angles are incommensurate in the circle, but in the triangle they are as simple as the angles 120° and 60°; for while the cosines of the latter are $\mp \frac{1}{2}$ the former angles and their supplements (approximately 109° and 71°) have cosines exactly $\mp \frac{1}{3}$; and as the trigonometrical measurement of angles is the most easy and natural, these angles are as easily constructed as the others.

2. But why, it may be asked, should the bees' instinct prefer a structure, in which all the angles of the planes are 120°, to any other, unless this structure be also the most economical? That it is the most symmetrical structure of cells that can be imagined will be granted, in spite of its seeming complexity; but of what advantage is elegant symmetry to the bee, unless it also economizes labor and material? And what therefore could have fashioned the instinct of the bee except a supersensible principle of rational foresight, superior to mere sensible perception? In considering these questions let us look at the hive-cell from another point of view. Having built it up *a priori* from general principles, let us see in what relations it stands *a posteriori* to other cells.

We learn from naturalists that the cell is a species of nest or open cocoon, and that cocoons and nests are of two kinds, excavations and structures; but both kinds have these general characteristics of form. If closed, they are either spheres or cylindrical

figures with hemispherical ends; and if open, they are either hemispheres or cylindrical figures open at one end and terminated at the other by hemispheres. The latter form of cocoon, nest, or cell is then the natural type of the honey-cell. When finished the cells resemble a series of excavations in wax, though in fact they are structures. That a single excavation in sand or wood — a cylindrical pit terminated spherically — has even less surface for the same contents than the honey-cell, has never excited remark on account perhaps of other and more obvious utilities in such a simple symmetrical form.

If a series of such excavations be made as closely together as possible, there would result an arrangement like that of a pile of equal cylinders, each pit surrounded by six other pits, and though the surface of each cell would be enlarged in a greater proportion than the enclosed space by changing the figures of the excavations, still space would be gained, and, in case of a structure, material would be saved, by converting the cylindrical cells into regular hexagonal prisms, (as may be seen in Fig. 2); and this too by simple saving, or by the economy of afterthought. Again, if equal spherical excavations be made as closely together as possible there would result an arrangement like that of a pile of equal spheres (Fig. 2), each cavity surrounded by six others in the same layer, and by three others in each of the contiguous layers, — in all, by twelve spherical cavities. The walls that would be left by removing

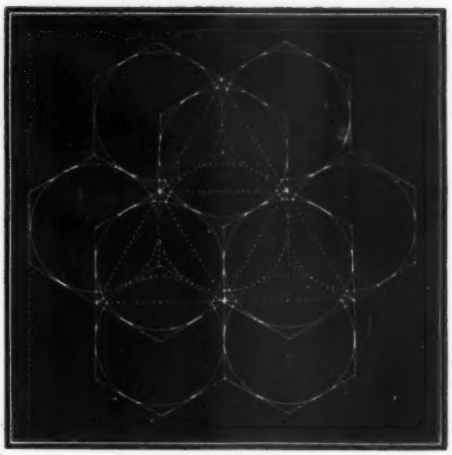


Fig. 2.

the superfluous material of the interspaces or corners would bound regular hexagonal prisms, terminated at both ends by pyramidal bases (Figs. 4, 5), each of which would be composed of three rhombs of the same form as the rhombs of the honey-cell. If, therefore, we were to put into the cells of the honey-comb little spheres which would just fit them, and should press them to the bottoms of the cells on both sides of the comb, the spheres would touch the middle points of the rhombs and the middle lines of the sides, so that the sides and bases of the cells would be tangent planes to those points at which the spheres would touch each other but for the thickness of the intervening walls.

To prove this, let us consider two spheres in contact in two contiguous layers of a pile. It will be seen from Fig. 2 that the projections of the centres of two such spheres upon the ground plane are distant from each other by the length of a side of the circumscribed hexagon. If, therefore, in Fig. 3 we draw two vertical parallel lines, at the distance $O\,M$ apart, equal to the side of the hexagon circumscribed around the circle O, then O and O' upon these two lines, at a distance apart of twice the radius of the circle, will represent the positions of the centres of the two spheres in contiguous layers. A, bisecting $O\,O'$, is the point of contact of the two spheres, and $B\,C$, perpendicular to $O\,O'$, is tangent to the two spheres, and is the shorter axis of the tangent rhomb. But $O\,O'$, equal to $P\,R$ or to twice the radius of the spheres,

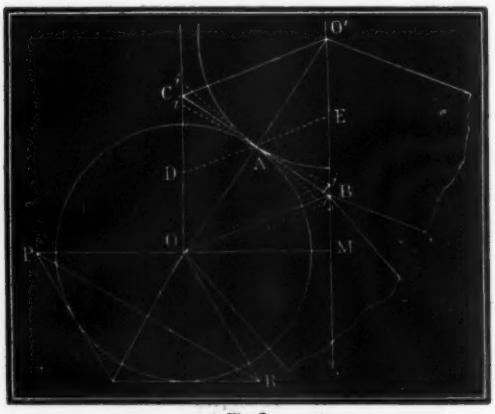


Fig. 3.

is equal to the longer axis of the rhomb. Hence by drawing $C\,O'$ and $B\,O$ we have a figure, $C\,O\,B\,O'$, of the same form as the tangent rhomb. If this figure were turned around its shorter axis, $B\,C$, to a position perpendicular to the plane of the diagram, it would coincide with the tangent rhomb. Its dimensions can be readily computed.

If r denote the radius of the spheres or circles, then the side of the circumscribed hexagon is obviously $r\sqrt{\frac{4}{3}}$, and as $O\,O' = 2\,r$ we find $O'\,M = r\sqrt{\frac{8}{3}}$. As the ratio of the axes of the rhomb is equal to the ratio of $O'\,M$ to $O\,M$ or to $\sqrt{\frac{8}{3}} \div \sqrt{\frac{4}{2}} = \sqrt{2}$, we find $B\,C = \frac{2r}{\sqrt{2}} = r\sqrt{2}$. Hence the sides of the rhomb are equal to $r\sqrt{\frac{3}{2}}$. $B\,M$ is therefore equal to $r(\sqrt{\frac{8}{3}} - \sqrt{\frac{3}{2}}) = r\sqrt{\frac{1}{6}}$, and the ratio of $B\,M$ to $B\,O$ is $\sqrt{\frac{1}{6}} \div \sqrt{\frac{3}{2}} = \frac{1}{3}$, which is therefore the cosine of the angles of the rhomb. Hence these angles are the same as the angles in the rhombs of the honey-cell. This may be shown still simpler, without computation, by observing after MARALDI, that the solid angles of the honey-cell are all regular, that is, composed of equal plane angles. The triangular ones, which are formed by the intersections of the three rhombs and by the intersection of each rhomb with two sides of the cell, have the common plane angle 109° 28′ nearly, while the quadrilateral solid angles formed by the intersection of two rhombs and two sides have the common plane angle 70° 32′ nearly (see Figs. 4, 5). Now as all the spheres which can be drawn tangent to all the sides of a regular solid angle must touch them in lines that bisect them, it follows that a sphere touching all the sides and rhombs of the honey-cell must touch the rhombs in the intersections of these bisecting lines or in their middle points, and the sides in their middle lines. In fine, the form of the honey-cell is that which would be obtained by placing equal spheres in two layers, as in Fig. 2, and drawing tangent planes through their points of contact, and terminating these planes in

their mutual intersections. These planes may be regarded, without reference to the contact of spheres, as planes midway between equidistant points in two parallel series. They would therefore pass through the intersections of such equal spheres as have radii greater than half the distance between their centres. Fig. 3 also represents portions of two sides of the cell turned up into the plane of the rhomb around the corners $O\,B$ and $O'\,B$. It also shows that the angle made by the plane of the rhomb, or by its shorter axis with the axis of the cell, is half the larger angle of the rhomb.

We have thus determined the form of the honey-cell from principles of symmetry alone. That this form is the most economical is easily shown by supposing it to vary, and by determining the changes of area in the sides and bases.

Let us suppose, as before, the rhomb $C\,O\,B\,O'$ to be turned upon the axis $B\,C$ by a right angle, so that the corners O and O' may fall in direction upon A, and let the sides that are attached be turned downward into their positions as vertical planes tangent to the sphere around O. In turning the rhomb now around its longer axis, the corners C and B must remain in the vertical lines $C\,O$ and $B\,O'$, so that the shorter axis is diminished if B is raised, and increased if B is depressed. Let i be the amount by which B is raised or depressed; then the projection of i upon the shorter axis $B\,C$ of the rhomb is half of the amount by which this axis is made shorter or longer. The projection of i upon $B\,C$ reduces it in the proportion of the semi-minor axis to the side of the rhomb or by the ratio $r\sqrt{\frac{1}{2}} \div r\sqrt{\frac{3}{2}}$, hence the projection is $i \times \sqrt{\frac{1}{3}}$. Half of this multiplied by the semi-major axis r is the change of area on each edge of the rhomb; hence the whole rhomb is decreased or increased by $2\,r\,.\,i\,.\sqrt{\frac{1}{3}}$. But $\frac{1}{2}\,i$ multiplied by the breadth of the side of the cell is the corresponding change in each of the sides attached to the rhomb, and hence both sides together are increased or decreased by

$i \times r\sqrt{\frac{4}{3}} = 2r.i.\sqrt{\frac{1}{3}}$. The sides gain as much therefore as the rhombs lose, or lose as much as the rhombs gain, by an infinitesimal change in their positions and form; and the whole surface is therefore either a *maximum* or a *minimum*, while the contents of the cell is unchanged by such a change of form.

Again, the infinitesimal triangles on either side of the line $B\,C$, resting on the bases i, are the amounts which the sides of the cell are made to gain or lose, while the smaller infinitesimal triangles on either side of $B\,C$, and resting on the projections of i, are the amounts which the edges of the rhombs lose or gain. These smaller triangles are in vanishing equal to half the larger ones, since the infinitely smaller right triangles resting upon i are similar to $O'A\,B$, the hypothenuse of which, $O'B$, homologous to i, is bisected by the line $D\,E$, and to this line the sides of the smaller infinitesimal triangles become parallel. If i be finite, the decrease of the minor axis of the rhomb will be less than twice the projection of i upon $B\,C$, and its increase will be more than twice this quantity; hence the rhomb will lose less than the sides gain, or gain more than the sides lose, by a finite change in the positions and form of the rhombs. The whole surface is therefore a *minimum*. Thus the symmetrical form of the honey-cell is seen to be also a *minimum* form.

This economy however has no reference to the depth of the cell, so that geometricians have conceived of a further limitation of the cell's form, which is not observed by the bee. Having determined the most economical arrangement of the sides and bases, we may further limit the length and breadth of the sides to that ratio, which gives what has been called and determined as the *minimum minimorum* form of the cell. If A denote the whole area of the cell's surface, C the whole solid contents, and l the length of the longer edges of the cell, or of those which terminate in the longer axes of the rhombs, (Figs. 4, 5,) and if, as before, we denote by r the radius of

the inscribed sphere or cylinder, then from the dimensions already computed we find

$$A = 2r^2\sqrt{2} + 4lr\sqrt{3}, \qquad C = 2r^2 l\sqrt{3}.$$

If one of these be supposed constant and the other a minimum, their derivatives must both be equal to zero. Hence by differentiation and reduction we have the equations

$$r\sqrt{2} + r\,D_r l\sqrt{3} + l\sqrt{3} = 0, \qquad 2l + r\,D_r l = 0,$$

from which, eliminating $D_r l$, we obtain $l = r\sqrt{\frac{2}{3}}$. Hence, l in the *minimum minimorum* cell is half the distance $O'M$ between the two planes on which the centres of the contiguous inscribed spheres are arranged. This form of the honey-cell is therefore what the bee would fashion if, instead of its deep nests, it should construct hemispherical nests like the bird, and then convert them into polyhedral figures with sides tangent to the contiguous hemispheres on both sides of the comb. Two such cells would exactly enclose a sphere with tangent planes, as in Figs. 4, 5. Of all figures, therefore, which can divide space into equal and similar parts, without interstices, the polyhedrons of Figs. 4, 5 have the least surface for a given contents.

Fig. 4.

These figures are such as would be formed by drawing tangent planes through the twelve points of contact on a sphere surrounded by twelve equal spheres. The two forms arise from the two ways in which the three spheres, in each of the outside layers, may be arranged. It may be seen in Fig. 2 that the three superposed spheres occupy alternate interstices; so that there are two ways in which they may be placed in contact with the central one. If the three spheres, which are in contact with the central one from below, be opposite

the three superposed ones, we have the polyhedron of Fig. 5; if alternately disposed, we have that of Fig. 4, the sides of which are all equal and similar rhombs, like the rhombs of the honey-cell. Fig. 5 may be produced by turning the lower half of Fig. 4 one sixth of its circumference around its vertical axis.

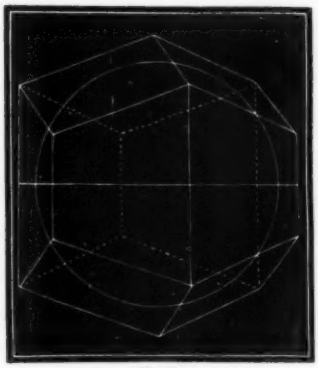


Fig. 5.

Geometricians have also criticised the economy of the bee by computing the difference between the surface of cells with plane bases and the surface of honey-cells having the same contents; and by showing that this difference is only about $\frac{1}{50}$ part of the whole surface of a completed cell; so that the bee is enabled by the pyramidal form of the base to make 51 completed cells, instead of 50, out of the same material, — a gain hardly worth so great precision in construction.

3. But this criticism would be just only on the suppositions, 1st, that a plane base could be more readily made than a pyramidal one, and 2dly, that the instinct of the bee is determined to a structure the most economical *a priori* and on the whole. These suppositions are not however necessary, nor are they supported by facts.

It is true, indeed, as we have seen, that the form of the honey-cell requires less material than any similar one for the construction of a double series of cells like the honeycomb. But we have also seen that simple nests are bounded by still less surface than the honey-cell, in proportion to their contents, and that the deviation of the single honey-cell from the cylindrical nest with a hemispherical base does in fact increase the surface in a greater proportion than it increases the enclosed space, — with a gain however of material when the cells are contiguous. As their bases are boundaries be-

tween cells on opposite sides of the comb, the pyramidal form would be possible as a modification of the hemisphere only in case of a close symmetrical arrangement of the opposite cells. But for such an arrangement, — of two series of spherical bases in closest contact, — a modification to the pyramidal form of the honey-cell is not merely the most economical, but also the simplest, and one perfectly analogous to the hexagonal modification of the sides.

That the cell ought to be regarded as such a modification of the simple nest by *simple* economy will be evident when we consider the mode of its construction. If the cells were excavated nests made in a costly material, they would by the closest arrangement and the removal of the interstitial material, receive through the agency of simple economy the hexagonal form of the honey-cell; but the bases of opposite cells could only by accident meet as they do in the honeycomb, and hence they would either retain, without reference to each other, the hemispherical form, or be flattened to plane hexagonal bases. But the hive-bee does in fact build up its nests from their bases which are rudely fashioned at first on the edges of the comb in the form of little cavities at equal distances. On one side of the comb these cavities form by their depressions the interstitial elevations between the cavities of the other side; and hence the symmetrical arrangement of the two series of cells, and hence also the pyramidal form of the bases and their economy.

It has been observed that mechanical pressure alone is capable of changing the elongated cells of cellular tissues — as in the fruits and piths of some plants — into the polyhedral form of the honey-cell. This however is not the way in which the honey-cell is formed, for the bees fashion the corners of their cells as they build, by continual trimming and saving.

Again, the bee builds with great precision; indeed, the exactness of the structure is more surprising than its economy. But precision

is not required by economy at those limits of form which determine *maxima* and *minima* values; for these values are determined by the condition that they shall vary least by slight changes in the forms on which they depend. Symmetry, on the contrary, requires absolute precision and affords the means of effecting it. Thus, when the bee's point of view is just over each one of the middle points of the rhombs it ought to be at the same distance from all the nine planes of the cell, and just opposite similar points of view in the nine contiguous cells. This symmetry affords a working guide to the bee as perfect as the regularity of the solid angles or the equality of all the angles made by the planes with each other.

It is true that these symmetries also determine the economy by which the honey-cell is characterized; so also does the still simpler symmetry of the cylindrical and hemispherical nest determine a still greater economy of work and surface in the architecture of innumerable other nest-building animals. But have psychologists, therefore, thought it necessary to regard the instincts of these animals as determined by supersensible properties of form, instead of that facility of construction and simple saving which are apparent to the senses?

Many deviations not only from the strictest economy, but also from symmetry, are observed in the honeycomb, and are required by the various sizes and uses of the cells.

In the normal form of the comb the triangular corners are formed by the meeting of four cells or by the intersection of six planes, three belonging to the bases and three to the sides; while the quadrilateral corners are formed by the meeting of six cells, or the intersection of twelve planes, six belonging to the bases and six to the sides. Now the bees very frequently in finishing their cells do not make these twelve planes meet in a common point, but connect them by a thirteenth very small plane, forming a tenth partition or

a fourth basal segment to two such cells on opposite sides of the comb as would otherwise only touch each other by a common corner. This supplementary basal segment is however perpendicular to the line which joins the symmetrical centres of the two cells, and is therefore easily included in the symmetries of the cell by the following considerations.

We have seen that if the lines joining equidistant points in two parallel planes be bisected by planes perpendicular to them, these planes terminating in their mutual intersections will form the corners of the honey-comb. Such lines of juncture form the edges of the regular tetrahedrons and octohedrons into which the space between the two planes may be divided; each tetrahedron surrounded by three octohedrons, and each octohedron by six tetrahedrons. The six planes that bisect the six edges of each tetrahedron meet in the centre of the figure and form four contiguous triangular corners. The twelve planes that bisect the twelve edges of each octohedron form in the centre of the figure six contiguous quadrilateral corners; and if in addition to these planes three be drawn midway between each pair of diagonal corners in the octohedron, they will also pass through the centre of the figure, but would be of no service as partition walls between these corners in a perfectly regular figure. If, however, the points in the parallel planes be not exactly equidistant, and the included figure vary from a regular octohedron, the twelve planes cannot at the same time bisect the edges of the figure and meet in the centre; and the bee accordingly, forced by this necessity, instead of making these planes meet in a common point, chooses to join them by another small one midway between two diagonal corners of the figure. Thus the bees separate the central points of their nests by planes midway between the centres of contiguous nests on the same side and on opposite sides of the comb, and also in irregular structures, by smaller planes

between the centres of such nests on opposite sides of the comb, as would otherwise only touch each other in a common quadrilateral corner. This way of remedying irregularities of structure shows conclusively that the bee's instinct is chiefly governed by *symmetries of partition* instead of those that belong to any particular figure.

Though it is to be presumed from what we have seen, that special facilities and economies of labor and material govern all abnormal forms, yet habit, undoubtedly, must greatly modify them from special adaptations to general conformities of structure.

The symmetries which we have studied are obviously not the same to the bee and to the geometrician, for what the latter comprehends abstractly as symmetry of form, is only perceptible to the former concretely, as facility of construction. Like the two kinds of economy, the one is rational, the other sensible.

An unreflective and unforeseeing economy, which, without reference to an end, simply saves, through sensuous preference, what the conditions of life render useful and costly to the race, characterizes the whole animal kingdom. For even those animals which do not store food or build structures for themselves or their offspring, still select through sensuous preference the food that is nutritious and appropriate to them, without foreseeing its use. An application of such a simple saving to the typical structure of the nest results in the honey-cell. The bee's instinct ought not therefore to be regarded as an exception to animal instincts in general; much less ought animal instincts in general to be so interpreted as to include a misconception of the bee's instinct.

A natural choice from simple feelings and from heritable sensuous associations — empirical in form, whatever their origin — is all that is required to account for the most perfect work of instinct.

A SECOND BOOK IN GEOMETRY.

[Continued from Page 143.]

CHAPTER VIII.

THE MAXIMUM AREA.

114. I will prove only one more proposition; but I will select a difficult one, in order that it may require a number of preliminary proofs. I will select the proposition given in the "First Lessons in Geometry," chap. xxiii. § 14: — "Of all isoperimetrical figures the circle is the very largest."

115. When we attempt to analyze this, we shall see that it implies that any regular polygon is less than a circle isoperimetrical with it, and that any other polygon is less than a regular one, isoperimetrical with it.

116. Let us begin, however, by defining a few of the words we shall need to use.

117. A polygon is a plane figure bounded by straight lines.

118. The perimeter of a polygon is the sum of the length of its sides.

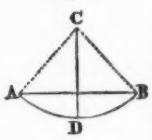
119. Isoperimetrical polygons are those of equal perimeter.

120. Among quantities of the same kind, the largest is called a maximum.

121. A circle is a plane figure, bounded by one line that curves equally in every part. This line is called the circumference of the circle, and frequently the line itself is called the circle. Portions of the circumference are called arcs.

122. *Theorem.* There is a point within the circle equally distant from every point of the circumference. This point is called the centre of the circle. — *Proof.* Let A D and B D be equal adjacent arcs in a circumference. Through the points A, B, and D draw lines at right angles to the curve at those points. Now, since the circle curves uniformly at every point, the point of intersection with B C must be the same point in both lines, A C and D C, and the points A, B, and D are equally distant from C. But A and B may be taken anywhere in the circle, only provided they are equally distant from D; and hence every point in the circle is equally distant from C, the centre of the circle.

Fig. 60.



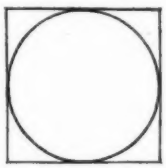
123. A straight line joining the centre to the circumference is called a radius. The straight line formed of two opposite radii is called a diameter.

124. All radii are of course equal to each other.

125. A straight line joining the two ends of an arc is called a chord.

126. A straight line which, however much prolonged, touches the circle in one point only, is called a tangent.

127. It is manifest that the tangent coincides in direction with the arc at the point of contact.



128. A polygon formed wholly of chords in a circle is said to be inscribed in that circle.

129. A polygon formed wholly by tangents to a circle is said to be circumscribed about the circle.

130. The circle is said to be inscribed in the circumscribed polygon, and to be circumscribed about the inscribed polygon.

131. If a polygon, about which a circle can be circumscribed, or in which a circle can be inscribed, has its sides equal, one to the other, the polygon is called a regular polygon, and the centre of these circles is called also the centre of the polygon.

132. Let us now attempt to analyze the proposition that the circle is the maximum among isoperimetrical polygons. This is equivalent to saying that if an isoperimetrical circle and regular polygon are laid one over the other, the polygon will be the smaller. But we see that by laying them one on the other, a circle inscribed in the polygon would be smaller than the isoperimetrical circle. The question of course suggests itself, whether the area of a regular polygon is not proportional to the radius of the inscribed circle. Now it is plain that it is. For by dividing the polygon into triangles, by lines from its centre to its vertices, we find the area of the polygon will be the sum of the areas of the triangles, and these areas will be measured by half the product of the perimeter multiplied by the radius of the inscribed circle. The area of the isoperimetrical circle will be measured by half the product of the perimeter multiplied by the radius. But as the perimeters of the polygon and the isoperimetrical circle are the same, and the radius of the inscribed circle is smaller than that of the isoperimetrical circle, it is evident that the area of the polygon is smaller than that of the isoperimetrical circle.

It will now remain to show that the area of a regular polygon is greater than that of an isoperimetrical irregular polygon. It is evident that this can be done, since a polygon of given sides is manifestly largest when most nearly circular, and a polygon of given perimeter inscribed in a circle is manifestly largest when the sides are equal. We can surely have no difficulty in proving these two points, and then our proof will be complete.

133. Let us return, then, to the synthetic mode, and establish these propositions: — First, that the maximum of polygons formed of given sides may be inscribed in a circle; secondly, that the maximum of isoperimetrical polygons having a given number of sides has its sides equal; and thirdly, that such a regular polygon is of smaller area than a circle isoperimetrical with it.

134. *Theorem.* The area of a triangle is found by multiplying the base by half the altitude. This theorem has been already proved (Art. 111).

135. We shall need the Pythagorean proposition, which implies all the propositions into which we have already analyzed it (Arts. 64 – 113.)

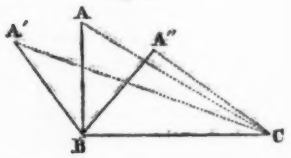
136. *Theorem.* Of two unequal lines, from a point to a third straight line, the shorter is more nearly perpendicular to the third line. — *Proof.* Let C be the given point, and A D the third straight line. Let C A and C B be two lines, of which C B is the shorter. Draw C D perpendicular to A D. We wish to prove that B D is shorter than A D. But this is manifest from the Pythagorean proposition, since the square on A D is the difference of the squares on A C and C D, and the square on B D is the difference of the squares on B C (which is smaller than A C) and the same C D.

137. *Corollary.* A perpendicular is the shortest line from a point to a given straight line.

138. *Theorem.* The maximum of triangles having two sides given is formed when these two sides are at right angles. — *Proof.* Let A′ B, A B, and A″ B be equal to each other. The area of A′ B C, A B C, or A″ B C, being found by multiplying B C into the perpendicular height

of A, A′, or A″, above B C, will be in proportion to that height. (Fig. 17.) Let then A B be perpendicular to B C, and the height of A above the base will equal B A. But the height of A″ above the base must, by 137, be less than B A″ which is equal to B A.

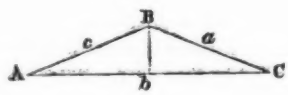
Fig. 17.



139. An angle is said to be measured by an arc of a circle such as would be intercepted by radii making that angle with each other. And, since the circumference curves equally in all parts, and the radii are at right angles to it, it is evident that this measure is just, and that the angle will bear the same ratio to four right angles that the arc bears to a whole circumference, whatever be the size of the circle.

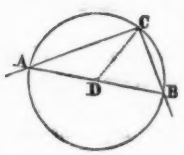
140. *Theorem.* If two sides in a triangle are equal, the angles opposite those sides are equal. — *Proof.* Let A B and B C be equal sides in a triangle. Imagine A C divided in the centre, at the point *b*. The triangles A B *b* and C B *b* will now be composed of equal sides, and we have already proved (Arts. 91 – 95) that they must have equal angles, — that is, the angle at A is equal to that at C.

Fig. 16.



141 *Theorem.* If one side of a triangle is prolonged, the external angle is equal to the sum of the opposite internal angles. This has been proved in Art. 57.

142 Two chords starting from one point in a circumference intercept double the arc that would be intercepted by radii making the same angle; or, the angle of the chords is measured by half the arc included between them. — *Proof.* If one chord, as A B, passes through the centre D of the circle, it is plain that by drawing D C the angle C D B will be equal to the sum of the angles C A D and D C A. But since D A and D C are equal, these angles are equal, and C D B is equal to twice C A D.

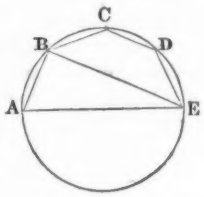


If neither chord passes through the centre of the circle, we can draw a third chord, starting from A, passing through the centre of the circle, and apply this reasoning to the two angles formed with this third chord by the other two. The angle of the other two chords will simply be the sum or the difference of these two angles.

143. *Corollary.* If the vertex of a right angle be placed in the circumference, the sides will intercept a semicircle.

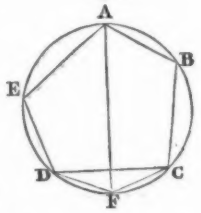
144. *Corollary.* If a circle be circumscribed about a triangle, and one side of the triangle passes through the centre of the circle, the opposite angle is a right angle.

145 *Theorem.* The maximum of polygons, having all the sides given but one, may have a circle circumscribed about it, having the unknown side for a diameter — *Proof.* Let A B C D E be the maximum polygon, formed of given sides, A B, B C, &c., and the unknown side, A E. Join B E by a straight line. Now, since the polygon is a maximum, we cannot, leaving B E unaltered, enlarge the triangle A B E, because that would enlarge the polygon. The angle A B E is therefore a right angle, by Art. 138, and a circumference, having A E for its diameter, would pass through the point B. In like manner it can be shown that a circumference having the same diameter would pass through each of the other points.



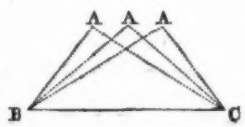
146. *Theorem.* The maximum of polygons formed with given sides can be inscribed in a circle. — *Proof.* Let A B C D E be a polygon formed of given sides, with a circle circumscribed about it. Draw the diameter A F, and join F C and F D. The polygons A B C F and A D E F are now maximum polygons, and therefore A B C D E must also be a maximum, since its enlargement would enlarge the sum of the other two.

Fig 9.

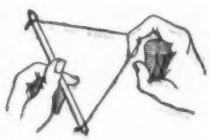


We have thus proved the converse of the proposition, and the proposition is true, unless there is more than one maximum form of the polygon.

The converse is more easily proved than the proposition, and I therefore proved it, on the assumption that there is but one maximum form. That is, I have proved that a polygon of given sides, when inscribed in a circle, is a maximum; but that does not strictly *prove* that the maximum can always be inscribed in a circle; except on the assumption, which is, however, a safe one, that a polygon formed of given sides, arranged in a given order of succession, can have but one maximum form.



147. *Theorem.* Of isoperimetrical triangles with one side given, the maximum has the two undetermined sides equal. — *Proof.* In order to prove this we have only to show that the point A is at its greatest distance from the base, B C, when opposite the middle of it. This might seem scarcely to need proof. For when we use a string and stick to illustrate the problem, we can see that by sliding the finger from the middle of the string, it can be brought down into a line with the stick; and the greatest height from the stick is near the middle of the string. Further consideration shows it must be exactly at the centre of the string, because the finger and string have precisely the same relation to one end of the stick as to the other; and a motion towards either end must affect the height of the finger in a similar manner.

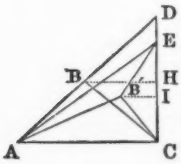


This reasoning is doubtless satisfactory to every fair mind. Yet it is not a good mathematical demonstration, and I have given it to you for the purpose of illustrating the peculiar nature of mathematical reasoning. The reasoning just given leaves no real doubt on the mind, but it is rather because we *see* with the eye that the finger is highest in the middle, than because we see with the mind that it must be. There is another step still lacking, to prove to us that the highest points are not on each side of the exact middle, as that would satisfy the conditions of symmetry and of declination towards each end. Let us then seek a proof which shall not force us to consider the whole motion of the finger, but which shall simply compare two forms of the triangle, one with the finger in the middle of the string, and one with the finger on one side.

148. *Theorem.* If a straight line be drawn from the vertex of two equal sides in a triangle, at right angles to the third side, it divides the third side into equal parts. — *Proof.* Let c and a be equal sides in a triangle A B C. Since (Fig. 16, Art. 140) the angles at A and C are equal, the angles b B C and b B A are also equal. If, therefore, the triangle B b c be folded over on the line B b, the line a will take the same direction as the line c, and, being of the same length, will coincide with it. Hence, b C will also coincide with b A, and the two lines must be of equal length.

149. New proof of Art. 147. Let A B C and A B′C be isoperimetrical, and let A B and

B C be equal. Continue A B to D, making B D = B A = B C, and join D C. Then, by Art. 144, the angle D C A is a right angle. Draw B′ E making it equal to B′ C. Join A E. A E will be less than the sum of A B′ and B′ E, that is less than A B′ and B′ C, that is less than A B and B C, that is less than A D. But if A E is less than A D, then C E must be less than C D, because the square on C E is equivalent to the difference of the squares on A E and A C, while the square on C D is equivalent to the difference of the squares on A D and A C. Draw B H and B′ I at right angles to C D; we have C I which is half C E less than C H which is half C D. But C I and C H are the altitudes of the triangles A B C and A B′ C above their common base A C. The triangle with the undetermined sides equal has the greatest altitude and must be the largest triangle.



150. *Theorem.* The maximum of isoperimetrical polygons of a given *number* of sides is equilateral, that is, has equal sides. — *Proof.* Let A B C E D (Fig. 9) be the maximum of isoperimetrical polygons of a given number of sides. Then A B must equal B C. For if it did not, then after joining A and C we could enlarge the triangle A B C by equalizing A B and A C, and thus enlarge the polygon without altering the number of sides or the perimeter, and the present form would not be the maximum.

In like manner we may prove that B C = C E, &c.

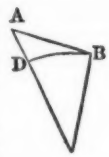
151. *Corollary.* The maximum of isoperimetrical polygons of a given number of sides is regular by Arts. 150 and 146.

152. *Axiom.* A circle may be considered as a regular polygon having an unlimited number of sides. And this regular polygon may be considered as either inscribed in or circumscribed about the real curve.

153. *Theorem.* The area of a regular polygon is measured by half the product of the perimeter into the radius of the inscribed circle. — *Proof.* For if lines be drawn from each vertex of the polygon to the centre, the polygon will be divided into triangles having a common altitude equal to the radius of the inscribed circle, the sum of the bases of these triangles being equal to the perimeter of the polygon.

154. *Corollary.* The area of a circle is measured by half the product of the radius into the circumference.

155. *Theorem.* The perimeter of a circumscribed polygon is greater than the circumference of the circle. — *Proof.* Let A B be half a side of a circumscribed polygon, and D B the portion of arc intercepted by lines drawn from A and B to the centre of the circle. Divide D B into arcs so small that each may be considered as a short straight line. Through the points of division draw lines extending from the line A B to the point C. At the end B, the little arcs are equal to the corresponding pieces of the line A B; but as you approach A the divisions of the line grow longer than the corresponding divisions of the arc, for two reasons; first, the little arcs are at right angles to the radii, while the portions of the line are not (Art. 136); secondly, the little arcs are nearer to the point C, towards which the radii converge. The whole of A B must therefore be longer than the whole of D B. But it is manifest that the circumference consists of as many times D B as the perimeter does of the line A B.



156. *Corollary.* The circle inscribed in a regular polygon is smaller than a circle isoperimetrical with the polygon and has a shorter radius.

157. *Corollary.* The circle is the maximum among isoperimetrical regular polygons.

158. *Corollary.* The circle is the maximum among isoperimetrical figures; a proposition

towards which we have been directing our course through 48 articles, some of which are themselves complex propositions referring to the preceding chapters. No other science requires anything like such long trains of connected reasoning, as those used in the mathematics. An argument in other matters usually consists of only a few steps; — what are called long arguments being really a collection of shorter independent proofs of the same thing. In the mathematics we are frequently required to take, as in the present instance, hundreds of consecutive steps to attain a single position.

159. *Scholium.* A slight modification of the reasoning in Arts. 155 – 157 would show that of isoperimetrical polygons that is greatest which has the greatest number of sides.

160. He that really wishes to learn geometry must learn to work alone. I advise the learner now to take up "First Lessons in Geometry," and, beginning with the fourth chapter, go through to the twenty-sixth, trying how many of the facts he can prove. I think he can, if he sets himself to work with a good will, prove the greater part. Perhaps he will be obliged to ask some help of his teacher, but I think not much. He will, however, do well to show his demonstrations to his teacher for his criticism.

When he comes to Chap. XXVI. of the "First Lessons" he will be obliged to lay down the book again, as the propositions in the remainder of the book cannot be proved without the aid of higher branches of mathematics, — Algebra, Trigonometry, and the Calculus.

161. In proving the Pythagorean proposition, and the proposition that the circle is the maximum among isoperimetrical plane figures, I have tried to give good examples of the mathematical mode of proof; — the analysis, in which the mind turns the proposition over in every form, trying all sorts of experiments upon it intellectually, to discover its vulnerable side; — and the synthesis, by which we then enter step by step into the very secret of the mystery.

Analysis consists in taking the proposition itself as the starting point, and going, step by step, to self-evident truths, or at least to truths already proved. Synthesis consists in starting with self-evident truths, or truths already proved, and going step by step to the truth which you would prove. But synthesis generally requires a previous rough analysis by which you select the proper point of departure for your synthetical reasoning.

A species of analysis called *reductio ad absurdum* is often used, in cases where true analysis, or true synthesis, is difficult. In this form of proof you assume that your proposition is not true, and by analysis show that this would lead you, step by step, to the denial of self-evident truth. This shows the proposition to be true, by simply showing that it cannot be false. Article 138 gives an instance of this proof.

I think you will, after mastering this book thoroughly, be able to read any of the books on geometry which you will be at all likely to meet with.

END OF PART I.

Mathematical Monthly Notices.

The Ladies and Gentlemen's Diary, or United States Almanac, and Repository of Science and Amusement. Intended for an Annual Magazine, including a variety of matter, chiefly original, on subjects of general utility, in the Arts, Sciences, Agriculture, Manufactures, &c., &c. Edited by M. Nash. New York: J. Seymour, Printer. 1819.

The work is divided into two parts. The former entitled the Ladies and Gentlemen's Diary,

or United States Almanac, is designed, as stated in the address to the public, "to provide for the friends of science a convenient guide and companion, which will assist those already skilled in the science of Astronomy, and instruct those who have had less opportunities for information. To offer superior advantages to ingenious persons of every description, particularly to such as are employed in coasting vessels, for an improved knowledge of the tides, and to whom a greater acquaintance with the stars must not only be agreeable, but of eminent utility — to be in many respects a national work — an Almanack for the United States. The latter, entitled the Repository of Science and Amusement, "will include original essays, relating to the arts and sciences; and every year mathematical questions to be answered in the next. Then will follow literary and philosophic queries, poetical enigmas, rebusses, and charades; likewise to be answered in the succeeding year."

The mathematical department was mainly enriched by the contributions of Prof. ADRAIN, JOHN MACAULEY, of Liverpool, England, the Hon. WALTER FOLGER, JR., Member of Congress from Nantucket, GEORGE GARTH, Philadelphia, JOHN CAPP, Harrisburgh, Penn., and JAMES PHILLIPS, Haerlem, N. Y., now Prof. of Mathematics in the University of North Carolina. GEORGE GARTH, of Philadelphia, was by far the most successful competitor in the solution of the Philosophic Queries.

Three numbers only of this highly interesting and instructive work were published. The first number appeared in 1820. — WILLIAM TIMPSON, White Plains, N. Y.

An Elementary Treatise on the Lunar Theory, with a brief sketch of the history of the problem up to the time of NEWTON. By HUGH GODFRAY, M. A., of St. John's College, Cambridge. Second edition, revised. MacMillan and Co., Cambridge, and 23 Henrietta Street, Covent Garden, London. 1859. pp. 119.

This work is intended as introductory to the more elaborate treatises on the Lunar Theory, and we heartily recommend it to those students who wish to understand the general character of this important problem. The first chapter is devoted to the "Principle of Superposition of Small Motions, and "Attractions of Spherical Bodies." The problems of two and three bodies are considered in the second chapter; and in the third, fourth, and fifth chapters we find the rigorous differential equations of the Moon's motions, their integration to quantities of the second order, and the methods which may be employed for computing the numerical values of the coefficients. The sixth chapter gives the physical interpretation of the results, the seventh, a series of propositions connected with the processes in the Lunar Theory, and the eighth, the history of the lunar problem before NEWTON. This little volume is an admirable introduction to the study of this great problem.

A Treatise on Conic Sections; containing an account of some of the most important modern algebraic and geometric methods. By REV. GEORGE SALMON, A. M., Fellow and Tutor, Trinity College, Dublin. Third edition, revised and enlarged. London: Longman, Brown, Green, and Longmans. 1855.

An Elementary Treatise on Conic Sections and Algebraic Geometry; with a numerous collection of easy examples, progressively arranged. Especially designed for the use of schools and beginners. By G. HALE PUCKLE, M. A., St. John's College, Cambridge, Principal of Windemere College. Second edition, enlarged and improved. MacMillan and Co., Cambridge, and 23 Henrietta Street, Covent Garden, London. 1858.

We have brought these two works together for the reason that the latter is written upon the same plan as the former, and is essentially introductory to it; and therefore what we have to say applies equally to both. It is hardly to be supposed that two such admirable works as these before us should be entirely unknown in this country after so many years' publication;

but we have no reason to suppose that they are by any means so generally known as to render a brief notice unnecessary.

We have already called the attention of our readers to two of MR. SALMON'S books, "A Treatise on Higher Plane Curves," and "Lessons on Higher Algebra," and we believe we shall be doing mathematical students a good service by urging them to give the present volumes their special attention. For those who have already made some progress in the study of Analytic Geometry MR. SALMON'S work will be sufficient. One most valuable feature in the plan of these works is the collections of problems which immediately follow the discussion of each proposition, to test the student's understanding of the subject as fast as it is gone over; the former contains over 400, and the latter nearly as many. In MR. SALMON'S book many of the problems are solved, and to a much larger number are appended such remarks and suggestions as will enable the student to master them. MR. PUCKLE gives the answers to all the questions at the end of the volume. But excellent as these volumes are in this respect, and also in the arrangement of the propositions, and the elegance and clearness of their demonstrations, we only call attention to them with regard to a want which the text books on the subject in use in this country do not supply. These volumes, the former more fully, give a systematic and intelligible account of "the algebraic and geometrical methods which have been introduced into use of late years;" but upon which our own text books are silent. Perhaps we shall make our idea more clear by specifying some of the methods. In the first place there is an "abridged notation" which very much facilitates otherwise long and tedious algebraical computations. Thus, if $\alpha = 0$, and $\beta = 0$, are the equations of two straight lines, then $\alpha - \beta = 0$ is that of a line passing through the intersection of $\alpha = 0$, $\beta = 0$, bisecting the angle they make with each other. If, then, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, denote the three sides of a triangle, $\alpha - \beta = 0$, $\beta - \gamma = 0$, $\gamma - \alpha = 0$, denote the equations of the bisectors of the angles of this triangle, and since their sum vanishes identically, we have the proof that the bisectors of the angles of a triangle meet in a point. Again, if $S = 0$ and $S' = 0$ are the equations of two conics, then $S + k S' = 0$ is that of the conic passing through the four real or imaginary points in which $S = 0$ and $S' = 0$ intersect. If S' breaks up into two right lines $\alpha = 0, \beta = 0$, then $S + k \alpha \beta = 0$ is the conic passing through the four points in which α and β intersect S. If both S and S' break into two right lines each, then $\alpha\gamma + k\beta\delta = 0$ denotes the conic circumscribing the quadrilateral $(\alpha\, \beta\, \gamma\, \delta)$. But we have no room for further illustration of the facility with which such a notation can be applied, to render propositions, quite difficult by the usual methods, almost self-evident.

We must refer the student to the volumes themselves, especially the former, for the methods of "Reciprocal Polars," "Harmonic and Anharmonic properties of Conics," "Involution," "Projections," &c. Indeed, some of the chapters of these books contain a new language for those only acquainted with the text books on Analytic Geometry of fifteen or twenty years ago. Those teachers and students who are desirous of becoming acquainted with the modern methods will never regret having had their particular attention called to these volumes.

Théorie Général de l'Élimination. Par LE CHEVALIER FRANÇOIS FAÀ DE BRUNO, Docteur ès Sciences de la Faculté de Paris, Capitaine Honoraire de l'État Major Sadre, Professeur Libre a l'Université de Turin. Paris: Libraire Centrale des Sciences, Leiber et Faraguet, Rue de Seine-Saint-Germain. No. 13. 1859.

There have always been two different classes of workers in every department of science, the inventors, and those who come after and systematize and popularize all that is known in any particular department. And while the former have a more direct agency in the progress of science, it may well be doubted whether the latter by the broad culture which is the inevi-

table result of their labors, are not, after all and on the whole, its greatest promoters, by preparing the way for future discoveries.

In this volume the labors of BEZOUT, JACOBI, BETTI, LIOUVILLE, CAYLEY, SYLVESTER, BORCHARDT, &c., which relate to elimination, but scattered through the various Journals and Proceedings of Academies, are co-ordinated under a single point of view. The aid thus rendered the student, not more for what the work furnishes, than for the sources of information which it points out, cannot fail to make it valuable. Indeed, it is only by the publication of such compends of the different branches of the science that we can hope, in the shortest time, and with the least labor, to become acquainted with their present limits, and the directions in which the student is most likely to be rewarded by new discoveries.

The work is divided into three parts; the first gives the theory of elimination between two equations; the second, between three equations; and the third, the elimination of many variables between any number of equations.

Editorial Items.

WE have received the following solutions of the Prize Problems in the March Number of the Monthly: —

AMANDA M. BENNETT, Saline, Mich., Probs. I., II.
HARRIET S. HAZELTINE, Worcester, Mass., Probs. I., II.
GEORGE A. OSBORNE, JR, Lawrence Scientific School, Cambridge, Mass., Probs. III., IV., V.
B. F. STANLY, Seminary, Cazenovia, N. Y., Prob. II.
JOHN J. CARTER, Nunda Lit. Institute, N. Y., Prob. I.
WILLIAM HINCHCLIFFE, Barre Plains, Mass., Probs. I., II., III.
FRANK N. DEVEREUX, Boston, Prob. II.
DAVID TROWBRIDGE, Perry City, N. Y., Probs. I., II., III., IV., V.
HORACE C. SYLVESTER, Boston, Prob. II.
JAMES F. ROBERSON, Ind. Univ., Bloomington, Probs. I., II., III.
ISAAC H. TURRELL, Mt. Carmel, Ind., Probs. I., II., V.
CHARLES FISH, Patten, Me., Prob. V.
ASHER B. EVANS, Madison Univ., N. Y., Probs. III., IV., V.
M. C. COREY, Exeter, N. H., Prob. II.

Errata. — On page 100, equations (10), for $q^{\frac{m}{n}}\ q^n$ read $\left(q^{\frac{m}{n}}\right)^n$; for $q^m \div q^n$ read $\sqrt[n]{q^m}$.

Books Received. — *Elements of Chemical Physics.* By JOSIAH P. COOKE, JR., Erving Professor of Chemistry and Mineralogy in Harvard University. 8vo. pp. 739. Boston: Little, Brown, and Co. 1860. *Théorie Nouvelle Géométrique et Mécanique des Lignes a Double Courbure*, par PAUL SERRET, Docteur ès Sciences, Membre de la Société Philomathique. Paris: Mallet-Bachelièr, Imprimeur-Libraire du Bureau des Longitudes, &c. 1860.

June, 1860.

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In order to adapt the book to the purposes of instruction, it has been prepared on a strictly inductive method throughout, and so much of the subject-matter of mechanics has been given at the beginning of the volume as was necessary to secure this object. For the same reason, each chapter is followed by a large number of problems, which are calculated not only to test the knowledge of the student, but also to extend and apply the principles stated in the text. Care has been taken to include the latest results of science, and several of the subjects have been so far developed as to illustrate the methods of physical research. The appendix contains a large collection of tables, such as are of daily use in the chemical laboratory, and which will render the work valuable to the professional chemist.

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Prof. Agassiz, the distinguished naturalist, writes: — "I am so much pleased with the plan of 'Gray's Lessons,' I find it so admirably adapted to the first wants of the student of that science, the order in which the different subjects are introduced is so true to nature, and so well fitted to give a correct idea of the true character of the life of plants, that *I truly hope it may be generally adopted as a text-book wherever Botany is taught.*"

Prof. Pearson, of Union College, writes of the "Manual": — "It is beyond comparison, *the best Manual* in this country, and should find a place in every institution where Botany is studied."

Dr. William Tully, formerly Professor of Materia Medica in Yale College, writes: — "Having had varied experience as a teacher of Botany for about thirty years, to pupils of both sexes, *I am of the opinion that the botanical works of Professor Gray ought to supersede all others*, not only in our colleges, but also in all the schools in the country."

They also have the indorsement of Drs. Lindley and Hooker, of London; Prof. Tuckerman, of Amherst College; Prof. Torrey, U. S. Assay Office; Prof. Silliman, of Yale College; Profs. Guyot and Schanck, of Princeton; Prof. Holton, of Middlebury College; Prof. Chadbourne, Williams College; Dr. Darlington, of Philadelphia; Prof. Henry, Smithsonian Institution; and many others.

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Notices.

A work which for nearly half a century has kept unflinchingly on the even tenor of its way, with a constantly growing reputation, has no need of culling "*notices*" from the Press. A single extract from a notice of the November number of 1859, from the Boston Courier and National Intelligencer, will show the estimation in which it is held.

"We are not aware that this number, whose contents we have briefly sketched, is richer than usual. We believe it steadily supports the character it has earned, of being, what good authorities have pronounced, the best scientific periodical in the English language which covers an equal, or nearly equal field. Its editorial staff warrants its abilities. Besides its venerable founder, there is the younger Silliman, and Professor J. D. Dana, our most encyclopædic and ablest general naturalist, assisted by Agassiz, Gray, and Wolcott Gibbs, in their several departments."

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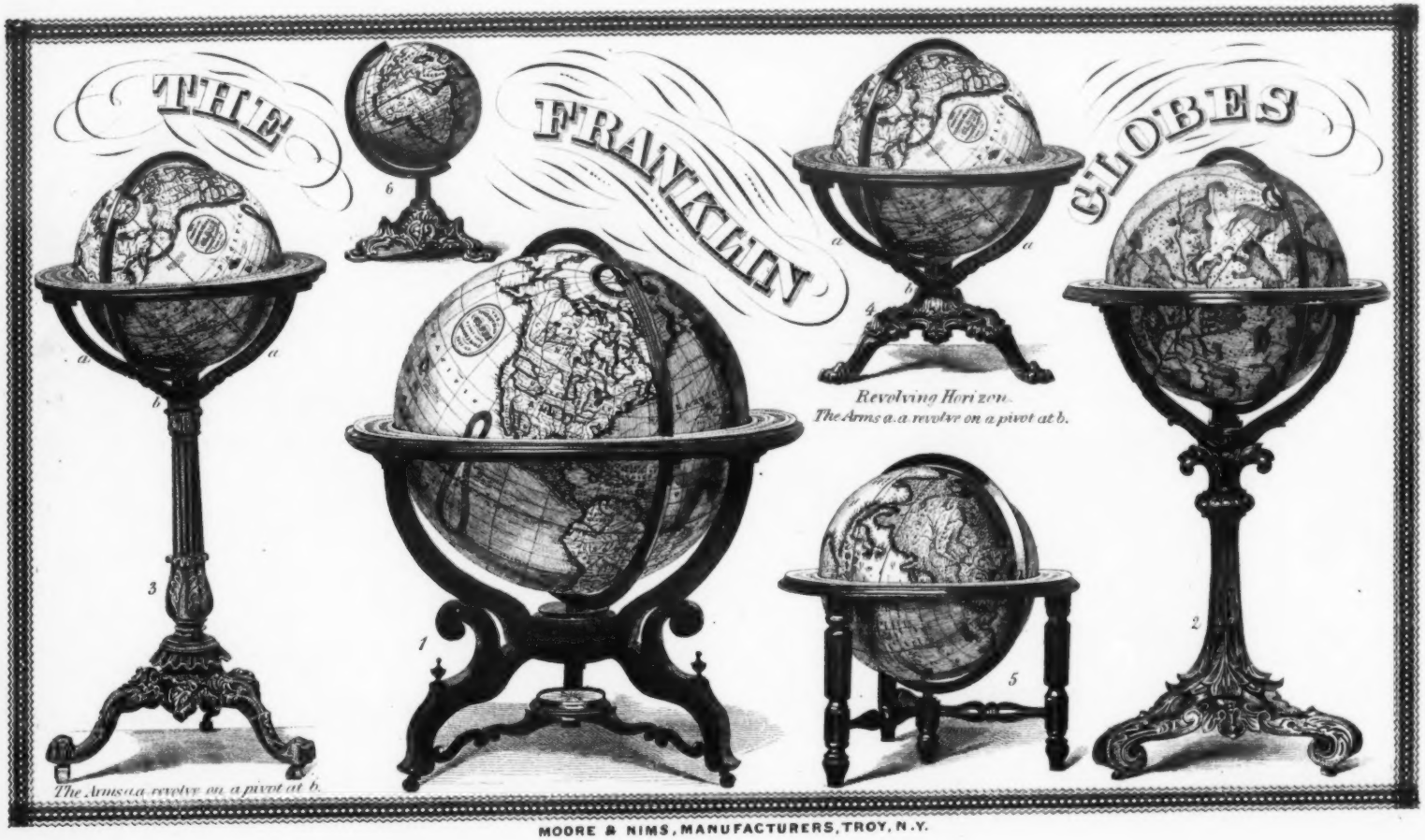

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